Supersymmetric transformations for the coupled-channel potentials and the inverse scattering problem

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Physical context and motivation

- Recent experimental progress in
  - cold atom-atom collisions
    (Bose-Einstein condensation, superfluids, atomic beams...)
  - low-energy nuclear collisions
    (nuclear astrophysics, radioactive beams, hypernuclei...)

  $\Rightarrow$ renewal of interest for low-energy quantum scattering theory
- Usual simplifications
  - two-body channels only
  - rotational invariance $\Rightarrow$ partial-wave decomposition
  - finite number of channels (close-coupling approximation)

  $\Rightarrow$ for each value of the total angular momentum (fixed J), small number N of coupled radial Schrödinger equations
- Not enough is known about the coupled-channel inverse scattering problem.

Coupled-channel radial Schrödinger equation

- System of N coupled equations (reduced units)

$$H\psi(k,r) = k^2\psi(k,r)\,, \quad H = -\frac{\partial^2}{\partial r^2} + V(r)\,, \quad V = \frac{l(l+1)}{r^2} + \Delta + \bar{V}(r),$$

  - $\bar{V}(r)$ – short-range symmetric $N \times N$ potential matrix,
  - $l = \mathrm{diag}(l_1, \ldots, l_N)$ – diagonal matrix of angular momenta,
  - $\Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_N)$ – diagonal matrix of thresholds,
  - $\psi = N \times 1$ or $N \times N$ solution matrix
- Three different types of the matrix Schrödinger equation
  - s-wave inelastic scattering, $\Delta \neq 0$, $l \equiv 0$,
    [Sparenberg *et al.*, JPA 2006]
  - elastic scattering, different partial waves, $\Delta \equiv 0$, $l \neq 0$,
    present work
  - inelastic scattering, different partial waves, $\Delta \neq 0$, $l \neq 0$
    ???$(\Delta \neq 0, l \neq 0) \sim (\Delta \neq 0, l = 0) \cup (\Delta = 0, l \neq 0)$

Jost matrix and scattering matrix

- Jost-solution $N \times N$ matrices $f(\pm k, r \to \infty) \sim \exp(\pm ikr)$ (basis)
- Regular-solution $N \times N$ matrix $\varphi(k, r)$,

$$\varphi(k, r \to 0) \sim r^{\nu+1}[(2\nu+1)!!]^{-1}\,, \quad V(r \to 0) = \frac{\nu(\nu+1)}{r^2}, \nu \geq l\,.$$

$$\varphi(k, r) = \frac{1}{2ik}\left[f(k, r)F(-k) - f(-k, r)F(k)\right]$$

- Jost matrix $F(k) = \lim_{r \to 0} \left[f^T(k, r) r^{\nu}\right][(2\nu-1)!!]^{-1}\,.$
- Scattering matrix $S(k)$ (unitary and symmetric)

$$\Psi(k, r) \underset{r \to \infty}{\propto} \exp(il\frac{\pi}{2} - ikr) + \exp(-il\frac{\pi}{2} + ikr)S(k)$$

$$\Rightarrow S(k) = \exp(il\frac{\pi}{2})F(-k)F^{-1}(k)\exp(il\frac{\pi}{2})$$

  - zero of det $F(k) \Rightarrow$ pole of $S(k)$ (bound, virtual or resonance states)
  - $F(k)$ and $S(k)$ non diagonal when $V(r)$ non diagonal (coupling)

Scattering matrix

Phase shifts, mixing parameter, $N = 2$

- S-matrix might be diagonalized

$$W^T(k) \begin{pmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{pmatrix} W(k) = \begin{pmatrix} e^{2i\delta_1(k)} & 0 \\ 0 & e^{2i\delta_2(k)} \end{pmatrix}, \qquad (1)$$

$$W(k) = \begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix}. \qquad (2)$$

  - $\delta_{1,2}$ – phase shifts (logarithm of the S-matrix's eigenvalues)
  - $\epsilon(k)$ – mixing parameter

Inverse scattering problem

Single-channel case

- Map: Scattering data (e.g., $\delta(k)$, $k \in (0, \infty)$ at fixed l) $\Rightarrow H$
  - Locality of the interaction potential $\bar{V}$
- General method - Integral equations
  - Gelfand-Levitan method: $S(k) \Rightarrow F(k) \Rightarrow$ spectral function $\rho(k) \Rightarrow$ input kernel $G(r', r) \Rightarrow$ output kernel $K(r', r) \Rightarrow V = 2\frac{d}{dr}K(r, r)$
  - Marchenko method: $S(k) \Rightarrow$ input kernel $A_0(r', r) \Rightarrow$ output kernel $A(r', r) \Rightarrow V = -2\frac{d}{dr}A(r, r)$
- Simplification for the case of the rational scattering matrix $S(k)$
- Bargman potential:
  $V_1 \to V_2 = V_1 - 2\frac{d}{dr}A(r, r) \Rightarrow F_2(k) = \prod_j \left(\frac{k+ia_j}{k+ib_j}\right) F_1(k)$
- Equivalence with SUSY transformation,
  [Sukumar C V , JPA 1985]

Supersymmetric-transformation principle: $V \to \tilde{V}$

$\to$

- $H\psi_E = E\psi_E$
- $\mathrm{spec}H$
- V
- ψ_E
- $F(k)$

$\xrightarrow{\mathcal{E},\sigma,U}$

- $\tilde{H}\tilde{\psi}_E = E\tilde{\psi}_E$
- $\mathrm{spec}\tilde{H} = \mathrm{spec}H, \pm\{\mathcal{E}\}$
- $\tilde{V} = V - 2\frac{d}{dr}U(r)$
- $\tilde{\psi}_E = L\psi$
- $\tilde{F}(k) = R(k)F(k)$

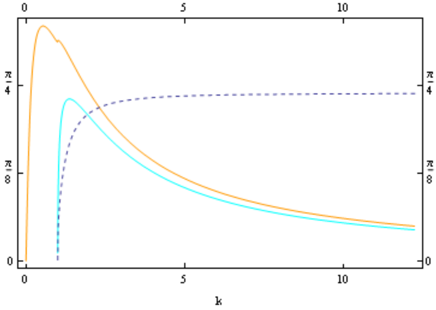
$\to$

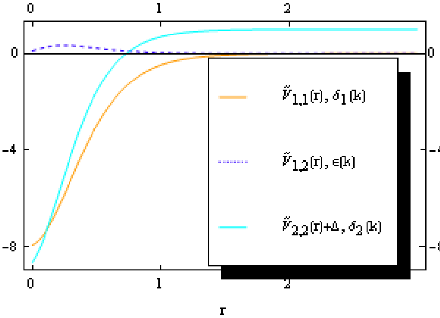
- SUSY transformation is determined by:
  - $\mathcal{E} = -\kappa^2$ – factorization constant; $H\sigma(r) = \mathcal{E}\sigma(r)$,
  - $\sigma(r)$ – transformation solution,
  - $U = \sigma'\sigma^{-1}$ – superpotential, prime$=\frac{d}{dr}$,
  - $L = -d/dr + U(r)$ – transformation operator,
  - $R(k) \equiv R[k; U(0), U(\infty)]$ – rational function, e.g. $k/(k - i\kappa)$,
  - Initial potential $V = 0 \Rightarrow$ simplest transformation solutions.
- Works for coupled channels when [Amado *et al.*, PRL 1988]
  - $V(r)$,$U(r)$,$\sigma(r)$, $R(k) = N \times N$ matrices, $N = 2$ below.

Coupled channel generalizations of ISM

Two-channel Cox potential, $\Delta \neq 0, l = 0$ [Cox, JMP 1964; Pupasov *et al.*, PRA 2008]

- Jost matrix $F_{\mathrm{Cox}}(k_1,k_2) = \begin{pmatrix} \frac{k_1+i\alpha_1}{k_1+i\kappa_1} & \frac{i\beta}{k_1+i\kappa_1} \\ \frac{i\beta}{k_2+i\kappa_2} & \frac{k_2+i\alpha_2}{k_2+i\kappa_2} \end{pmatrix}$,
- Revised by SUSY transformation, $V = 0 \Rightarrow \tilde{V}$,
  $\mathcal{E}I = \Delta - \mathcal{K}^2, \sigma = \cosh(\mathcal{K}r) + \mathcal{K}^{-1}\sinh(\mathcal{K}r)U(0)$,
- Parameters: $\alpha_1 = 0.15$, $\alpha_2 = 0.4$, $\beta = 0.1$ $\kappa_1 = 2$, $k_2 = \sqrt{k_1^2 - 1}$.



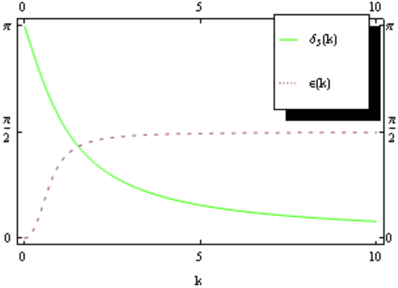


Coupled channel generalizations of ISM

Newton-Fulton potential $\Delta = 0, l = \mathrm{diag}(0,2)$, [Newton, Fulton, PRL 1957]

- The scattering matrix $S_{N-F}(k) =$

$$\frac{1}{k^4+4\chi^4}\begin{pmatrix} 2\chi^2 & k^2 \\ -k^2 & 2\chi^2 \end{pmatrix}\begin{pmatrix} \frac{(k+i\phi)(k+i\kappa)}{(k-i\phi)(k-i\kappa)} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2\chi^2 & -k^2 \\ k^2 & 2\chi^2 \end{pmatrix}.$$



- $\delta_s(k) = \arctan\frac{\phi}{k} + \arctan\frac{\chi}{k}$, $\delta_d(k) = 0$.
- $\epsilon(k) = \arctan\frac{k^2}{2\chi^2}$
- Parameters: $\phi = 1$, $\kappa = 1.5$, $\chi = 0.5$
- Gelfand-Levitan equation was generalized to find the potential

$\Rightarrow$ Challenge to the method of SUSY transformations!

Coupled channel SUSY transformations

Different partial waves without thresholds, $\Delta = 0, l \neq 0$

- Transformation function $\sigma_c(x) = f_d(-i\kappa, r)C_1 + f_d(i\kappa, r)C_2\,.$
  - $\det C_1, C_2 \neq 0 \Rightarrow$ diagonal $R(k)$, no coupling,
  - $\det C_1 = 0, \det C_2 \neq 0 \Rightarrow$ non-digonal $R(k)$, coupling,
- Mixing SUSY transformation $V_d \Rrightarrow V_c$
  - Parameters: x, q, κ

$$C_1 = \begin{pmatrix} 1 & 0 \\ q & 0 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} x & -q \\ 0 & 1 \end{pmatrix}. \tag{3}$$

  - Superpotential $U_c(r) = \sigma_c' \sigma_c^{-1}$, $U_c(\infty) = \lim_{r\to\infty} U_c(r)$

$$U_c(\infty) = \kappa \begin{pmatrix} \frac{1-q^2}{1+q^2} & \frac{2q}{1+q^2} \\ \frac{2q}{1+q^2} & \frac{q^2-1}{1+q^2} \end{pmatrix} = \kappa \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}, q = \tan\frac{\alpha}{2}$$

- Generalization of the Amado work: different partial waves, more general σ, detailed analysis of V_c and F_c

Coupled channel SUSY transformations

Properties of the mixing SUSY transformation

- Decreasing singularity of the potential at the origin

$$\nu_d = (\nu_1, \nu_2) \Rightarrow \nu_c = (\nu_1 - 1, \nu_2 - 1)\,,$$

- Long-range behavior of V_c
  - if $l_1 \neq l_2$: necessary to fix $q = 1$, $l_d = (l_1, l_2) \Rightarrow l_c = (l_2, l_1)\,,$
  - if $l_1 = l_2$: $l_d = (l_1, l_2) \Rightarrow l_c = (l_1, l_2)$ q - arbitrary.
- The Jost matrix transformation

$$F_c(k) = R(k)F_d(k)\,, R(k) = (k^2 + \kappa^2)(ikI - U_c(\infty))^{-1}\,,$$

- The Jost matrix determinant

$$\det F_c(k) = (k^2 + \kappa^2)\det F_d(k)\,,$$

- New bound, $k_b = i\kappa$, and virtual, $k_v = -i\kappa$, states.

Coupled channel SUSY transformations

The S-matrix transformation

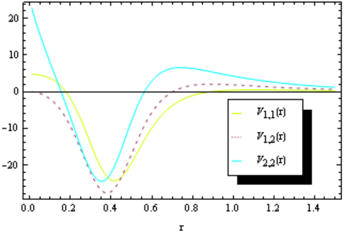
- Non-diagonal S-matrix $S_c(k) = -e^{il_c\pi/2}(ikI + U_{cinf})^{-1}e^{-il_d\pi/2}S_d(k)e^{-il_d\pi/2}(ikI - U_{cinf})e^{il_c\pi/2}$,
- mixing parameter $\epsilon(k)$, $\epsilon(k \to 0) \simeq k^{|l_2-l_1|}$,

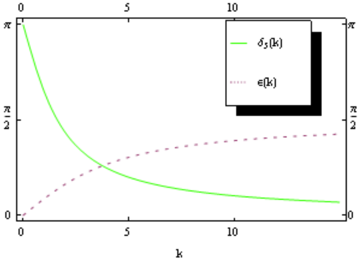
$$W^T(k)\begin{pmatrix} S_{c;11} & S_{c;12} \\ S_{c;12} & S_{c;22} \end{pmatrix} W(k) = \begin{pmatrix} e^{2i\delta_{c,1}(k)} & 0 \\ 0 & e^{2i\delta_{c,2}(k)} \end{pmatrix}, \quad (4)$$

$$W(k) = \begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix}. \quad (5)$$

  - unphysical case ("simple"): $l_2 = l_1 + 1(\mathrm{mod}2), q = 1 \Rightarrow \tan 2\epsilon_1 = \frac{2k\kappa}{\kappa^2-k^2}$, $\delta_d(k) = \delta_c(k)$,
  - physical case ("complicated"): $l_2 = l_1(\mathrm{mod}2), q = 1 \Rightarrow \tan 2\epsilon_2 = \frac{-2k\kappa}{(k^2+\kappa^2)}\cot\left(\delta_{d,2} - \delta_{d,1}\right)$.
  - zero energy bound state: $\epsilon(0) = 0 \Rightarrow \delta_{d,2}(0) - \delta_{d,1}(0) = \frac{\pi}{2}$.

SUSY partner of zero $s-p$ potential





- V_d is obtained by single-channel SUSY transformation: $\nu_d = \mathrm{diag}(2,1)$, $l_d = \mathrm{diag}(0,1)$
- Mixing transformation: $q=1, x=1, \kappa = 3.53$
- V_c: $\nu_c = \mathrm{diag}(1,0)$, $l_c = \mathrm{diag}(1,0)$
- Phase shifts $\delta_s(k) = \arctan\frac{s_1}{k} + \arctan\frac{s_2}{k}$, $\delta_p(k) = 0,\ s_1 = 1.5, s_2 = 1.75$,
- Mixing angle $\epsilon(k) = \arctan\frac{k}{\kappa}$,
- Compare with the Newton-Fulton S-matrix.

- Mixing transformation does not change the phase shifts!

Conclusions and perspectives

- Mixing transformation for N-channel uncoupled potential
- Solves schematic inverse problems for $N = 2$, $s - p$ coupling
- Future theoretical developments
  - iteration of mixing SUSY transformations (necessary to $s - d$ coupling)
- Future physical applications
  - neutron-proton interaction
  - low-energy nuclear collisions (Coulomb!)